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ENERGETICS OF A COLLISION WITH FRICTION†

A. P. IVANOV

Moscow

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The change in kinetic energy due to the collision of two rigid bodies is investigated. Different ways of defining the coefficient of restitution are discussed. It is shown that the Newton definition applied to an oblique impact can violate the law of conservation of energy. With the Poisson definition, the calculated energy dissipation is always positive, but slightly higher than the experimentally observed value. Energy definitions of the coefficient of restitution are more realistic.

1. STATEMENT OF THE PROBLEM

IN DYNAMICS, the problem of the collision of rigid bodies reduces to determining the collision impulse I with the following conditions: the coordinates of the system are unchanged by the collision, and the change in the velocities in an inertial system of coordinates can be described by the equations [1]

$$m_1 \Delta \mathbf{V}(G_1) = \mathbf{I}, \quad m_2 \Delta \mathbf{V}(G_2) = -\mathbf{I} \tag{1.1}$$

$$\mathbf{J}_1 \Delta \mathbf{W}_1 = \mathbf{r}_1 \times \mathbf{I}, \quad \mathbf{J}_2 \Delta \mathbf{W}_2 = -\mathbf{r}_2 \times \mathbf{I}, \quad \mathbf{r}_i = G_i A \quad (i = 1, 2)$$

where A is the point of contact of the colliding bodies, m_i and $\mathbf{J}_i$ are their mass and inertia tensors, $\mathbf{V}(G_i)$ are the velocities of the centres of mass of each body and $\mathbf{W}_i$ are the angular velocities.

We shall assume that the impulsive forces $\mathbf{F} = d\mathbf{I}/dt$ satisfy the Coulomb laws of friction. If $\mathbf{n}$ is the unit vector normal to the colliding surfaces at the point A, then the friction is determined by the coefficient μ and the direction of the relative velocity $\mathbf{V}_\tau$ at A:

$$\mathbf{V}_\tau = V - V_n \mathbf{n}, \quad V_n = (\mathbf{V}, \mathbf{n}), \quad \mathbf{V} = \mathbf{V}(G_1) + \mathbf{W}_1 \times \mathbf{r}_1 - \mathbf{V}(G_2) - \mathbf{W}_2 \times \mathbf{r}_2 \tag{1.2}$$

If $\mathbf{V}_\tau \neq 0$, we have

$$d\mathbf{I}_\tau / dI_n = -\mu \theta_\tau, \quad \theta_\tau = \mathbf{V}_\tau / |\mathbf{V}_\tau|, \quad I_n = (\mathbf{I}, \mathbf{n}), \quad \mathbf{I}_\tau = \mathbf{I} - I_n \mathbf{n} \tag{1.3}$$

but if $\mathbf{V}_\tau = 0$, $\mathbf{I}$ moves along the straight line of no slip (see [1]), defined in space $\mathbf{I} \in R^3$ by Eqs (1.1) and (1.2), with $\mathbf{V}_\tau = 0$.

System (1.1)–(1.3) can be used to construct the relation $\mathbf{I}(I_n)$, and all that is needed to solve the collision problem is to fix the value of I_n at the end of the collision. Newton's hypothesis of two collision phases is conventionally used for this [2]: the first phase, deformation, ends at a value $I_n = I_1$ for which $V_n = 0$. The second phase, restitution, ends at $I_n = I_2$. The coefficient of restitution is usually found in one of the following ways [1, 2]: either from the formula

$$I_2 = (1 + \kappa) I_1, \quad 0 \leqslant \kappa \leqslant 1 \tag{1.4}$$

or from the relation

$$V_n^+ = -e V_n^-, \quad 0 \leqslant e \leqslant 1 \tag{1.5}$$

where the indices minus and plus correspond to the beginning and end of the collision.

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In the general case $e \neq \kappa$, that is, definitions (1.4) and (1.5) are non-equivalent [3], although the incorrect view that they are does exist [4]. Each of the two models of restitution gives a unique definition of **I** for any initial conditions [1, 5, 6]. However, in itself this possibility is an insufficient basis for deciding on the suitability of a particular model.

Examples have been constructed [7] which show that (1.5) contradicts the law of conservation of energy. In this paper, we examine the possibility of correctly describing a collision with friction.

2. THE CHANGE IN ENERGY DURING A COLLISION

During a collision of rigid bodies, the change in their total kinetic energy is equal to the work of reaction **F** applied at the contact point A. The work done in moving $d\mathbf{r}$ has the form

$$dA = (\mathbf{F}, d\mathbf{r}) = (\mathbf{F}dt, d\mathbf{r}/dt) = (\mathbf{V}, d\mathbf{I}) \qquad (2.1)$$

The change in the kinetic energy of the system can be expressed by the curvilinear integral

$$\Delta T = \int_{0\mathbf{I}^*} (\mathbf{V}, d\mathbf{I}) \qquad (2.2)$$

where the integration is carried out from the value $I = 0$ to $I = \mathbf{I}^*$ of the impact impulse at the end of the collision.

There are certain results that follow from (2.2).

1. The integral (2.2) is independent of the shape of the path. In fact, relations (1.1) and (1.2) can be put in matrix form:

$$\Delta\mathbf{V} = \mathbf{C}\mathbf{I}, \qquad \mathbf{C} = (m_1^{-1} + m_2^{-1})\mathbf{E}_3 + \| c_{ij} \| \qquad (2.3)$$

$$c_{ij} = (\mathbf{J}_1^{-1}(\mathbf{r}_1 \times \mathbf{l}_i), \mathbf{r}_1 \times \mathbf{l}_j) + (\mathbf{J}_2^{-1}(\mathbf{r}_2 \times \mathbf{l}_i), \mathbf{r}_2 \times \mathbf{l}_j) = c_{ji}$$

$$\mathbf{l}_1 = (1, 0, 0), \quad \mathbf{l}_2 = (0, 1, 0), \quad \mathbf{l}_3 = (0, 0, 1).$$

where the matrix **C** is symmetric and positive definite. The integrand in (2.2) can be written in the form

$$(\mathbf{V}, d\mathbf{I}) = (\mathbf{V}^- + \mathbf{C}\mathbf{I}, d\mathbf{I}) = d((\mathbf{V}^-, \mathbf{I}) + 0{,}5(\mathbf{C}\mathbf{I}, \mathbf{I})) = dU(\mathbf{I}), \quad U(\mathbf{I}) = (\mathbf{V}^-, \mathbf{I}) + 0{,}5(\mathbf{C}\mathbf{I}, \mathbf{I}) \qquad (2.4)$$

from which the assertion follows.

2. Substituting (2.4) into (2.2), we have

$$\Delta T = U(\mathbf{I}^*) - U(\mathbf{0}) = (\mathbf{V}^-, \mathbf{I}) + 0{,}5(\mathbf{C}\mathbf{I}^*, \mathbf{I}^*) = 0{,}5(\mathbf{V}^- + \mathbf{V}^+, \mathbf{I}^*) \qquad (2.5)$$

Relation (2.5) is known as Kelvin's formula (see [1]). With its help, it is easy to find the value of ΔT if the impact impulse $\mathbf{I}^*$ is known. For the collision of two bodies in the general case, no explicit dependence of $\mathbf{I}^*$ on the initial conditions is known, and so Kelvin's formula does not give a result.

3. For friction described by (1.3), Eq. (2.2) becomes:

$$\Delta T = \int_0^{I_2} \left(V_n + \mathbf{V}_\tau \frac{dI_\tau}{dI_n} \right) dI_n = \int_0^{I_2} (V_n - \mu \, | \mathbf{V}_\tau |) dI_n \qquad (2.6)$$

where ΔT is expressed as a definite integral. Formula (2.6) also holds in the case when $V_\tau = 0$, since then $(\mathbf{V}, d\mathbf{I}) = V_n dI_n$.

4. Using the identity

$$\int_a^b f(x)\,dx = \int_a^b dx \left[f(0) + \int_0^x f'(y)\,dy\right]$$

(2.6) can be reduced to the form

$$\Delta T = \int_0^{I_2} dI_n \left[V_n^- - \mu \, | \mathbf{V}_\tau^- | + \int_0^{I_n} \left(\frac{dV_n}{dI_n} - \mu \frac{d\,|\mathbf{V}_\tau|}{dI_n}\right) dI_n \right. \qquad (2.7)$$

If $\mathbf{V}_\tau \neq 0$, the integrand in (2.7), taking (1.3) and (2.3) into account, can be transformed as follows:

$$\frac{d\mathbf{V}}{dI_n} = \frac{d(\mathbf{CI})}{dI_n} = \mathbf{C}\frac{d(I_n\mathbf{n} + \mathbf{I}_\tau)}{dI_n} = \mathbf{Cn} - \mu\mathbf{C}\theta_\tau = \mathbf{b} \tag{2.8}$$

$$\frac{dV_n}{dI_n} = (\frac{d\mathbf{V}}{dI_n}, \mathbf{n}) = (\mathbf{b}, \mathbf{n}), \quad \frac{d\mathbf{V}_\tau}{dI_n} = \frac{d\mathbf{V}}{dI_n} - \mathbf{n}\frac{dV_n}{dI_n} = \mathbf{b} - \mathbf{n}(\mathbf{b}, \mathbf{n}),$$

$$\frac{d\,|\mathbf{V}_\tau|}{dI_n} = \frac{d}{dI_n}\sqrt{(\mathbf{V}_\tau, \mathbf{V}_\tau)} = \frac{1}{|\mathbf{V}_\tau|}(\frac{d\mathbf{V}_\tau}{dI_n}, \mathbf{V}_\tau) = (\mathbf{b}, \theta_\tau),$$

$$\frac{d\theta_\tau}{dI_n} = \frac{d}{dI_n}(\frac{\mathbf{V}_\tau}{|\mathbf{V}_\tau|}) = \frac{1}{|\mathbf{V}_\tau|}(\frac{d\mathbf{V}_\tau}{dI_n} - \theta_\tau\frac{d\,|\mathbf{V}_\tau|}{dI_n}) =$$

$$= \frac{1}{|\mathbf{V}_\tau|}[\mathbf{b} - \mathbf{n}(\mathbf{b}, \mathbf{n}) - \theta_\tau(\mathbf{b}, \theta_\tau)]$$

$$\Phi(I_n) = \frac{dV_n}{dI_n} - \mu\frac{d\,|\mathbf{V}_\tau|}{dI_n} = (\mathbf{n} - \mu\theta_\tau, \mathbf{b}), \quad \mathbf{b} = C(\mathbf{n} - \mu\theta_\tau).$$

If $\mathbf{V}_\tau = 0$, only the first term in the inner integral in (2.7) is non-zero. The direction vector of the line of no slip $\mathbf{s}$ is defined by the equation $(\mathbf{Cs})_\tau = 0$, from which we obtain $\mathbf{s} = \mathbf{C}^{-1}\mathbf{n}$. When there is no slip, the following expression is obtained for $\Phi(I_n)$:

$$\Phi(I_n) = \frac{dV_n}{dI_n} = (\mathbf{Cn}, \mathbf{n}) + (\mathbf{C}\frac{d\mathbf{I}_\tau}{dI_n}, \mathbf{n}) = (\mathbf{Cn}, \mathbf{n}) + (\frac{\mathbf{s}_\tau}{s_n}, \mathbf{Cn}) = (\mathbf{Cn}, \mathbf{n}) + \tag{2.9}$$

$$+ (\mathbf{C}^{-1}\mathbf{n}, \mathbf{n})^{-1}(\mathbf{C}^{-1}\mathbf{n} - \mathbf{n}(\mathbf{C}^{-1}\mathbf{n}, \mathbf{n}), \mathbf{Cn}) = (\mathbf{C}^{-1}\mathbf{n}, \mathbf{n})^{-1}.$$

Taking (2.8) and (2.9) into account and the fact that $\mathbf{C}$ is positive definite, the following theorem has been proved.

Theorem 1. The increment of kinetic energy in a collision with Coulomb friction is given by the formula

$$\Delta T = \int_0^{I_2} Z(I_n)\,dI_n, Z(I_n) = V_n - \mu\,|\mathbf{V}_\tau| = Z(0) + \int_0^{I_n} \Phi(I_n)\,dI_n \tag{2.10}$$

where I_2 is the magnitude of the normal component of the impact impulse at the end of the collision. The function Z is negative when $I_n = 0$ and increases monotonically when $I_n > 0$ (since $Z' = \Phi > 0$).

3. ENERGY ANALYSIS OF THE CORRECTNESS OF THE TWO MODELS OF COLLISION

Since the work of the reaction forces is always non-positive, the energy criterion for the correctness of Eqs (1.1) has the form

$$\Delta T \leqslant 0. \tag{3.1}$$

We will verify inequality (3.1) for the two models of collision, described by Eqs (1.3) and (1.4), or (1.3) and (1.5).

1. If the vector $\mathbf{V}_\tau$ keeps its positive direction during the collision, $d\theta_\tau = 0$, then from (1.3) and (2.3), the value of V_n is a linear function of I_n. In that case $\kappa = e$, that is, descriptions (1.4) and (1.5) of the end of the collision are equivalent.

The function $Z(I_n)$ in (2.10) is a linear function of I_n:

$$Z(I_n) = Z(0) + I_n(\mathbf{n} - \mu\theta_\tau^-, \mathbf{C}(\mathbf{n} - \mu\theta_\tau^-))$$

Since $V_n = 0$ when $I_n = I_1$ we have $Z(I_1) \leqslant 0$. Hence

$$Z(I_2) = Z(I_1 + \kappa I_1) \leqslant -\kappa Z(0)$$

from which we obtain

$$\Delta T = \int_0^{I_1} Z(I_n) dI_n + \int_{I_1}^{I_2} Z(I_n) dI_n \leqslant 0 \tag{3.2}$$

2. If the vector $\mathbf{V}_\tau$ changes direction during the collision, model (1.3) and (1.5), with certain initial conditions, might lead to the paradoxical inequality $\Delta T > 0$ [7].

As an example, consider the two-dimensional collision of a rod of length $2L$ and the half-space, with which it makes an angle φ. System (1.1) takes the form

$$m\Delta\mathbf{V}(G) = \mathbf{I}, \quad J_3 \Delta W_3 = r_1 I_2 - r_2 I_1, \quad r_1 = L\cos\varphi, \quad r_2 = L\sin\varphi$$

The matrix $\mathbf{C}$ in (2.3) is such that

$$\mathbf{C} = \begin{Vmatrix} 1/m + r_1^2/J_3 & -r_1 r_2/J_3 \\ -r_1 r_2/J_3 & 1/m + r_2^2/J_3 \end{Vmatrix}.$$

If $L = 1$, $m = 1$, $J_3 = 1/6$, $\varphi = \pi/4$, $\mu = 1$, $e = 0.8$, $V_n^- = -1$, $V_\tau^- = 0.6$, then

$$\mathbf{C} = \begin{Vmatrix} 4 & -3 \\ -3 & 4 \end{Vmatrix}, \quad Z(0) = -1.6.$$

As I_n changes from 0 to the value $I^* = 0.086$, the tangential component of the velocity decreases to zero, in accordance with the equations

$$dI_\tau/dI_n = -1, \quad \Delta V_n = 4I_n - 3I_\tau = 7I_n, \quad \Delta V_\tau = -3I_n + 4I_\tau = -7I_n.$$

When $I_n > I^*$, slip is not renewed, and $d\mathbf{I}$ is parallel to the vector of no slip $\mathbf{s} = \mathbf{C}^{-1}\mathbf{n} = (3/7, 4/7)$, from which we obtain the following collision equations for the interval $I_n > I^*$:

$$dI_\tau/dI_n = 3/4, \quad I_\tau = 3/4 I_n - 7/4 I^*$$

$$\Delta V_n = 1.75 I_n + 5.25 I^*, \quad \Delta V_\tau = -V_\tau^-$$

We define the end of the collision from the condition (1.5): $e = -1 + 5.25I^* + 1.75I_2$, from which we obtain $I_2 = 0.77$. We find ΔT from (2.6):

$$\Delta T = \int_0^{I^*} (V_n^- + 7I_n - \mu \mid V_\tau^- - 7I_n \mid) dI_n + \int_{I^*}^{I_2} (V_n^- + 5.25I^* + 1.75I_n) dI_n = 0.051 > 0$$

This example shows that it is incorrect to use (1.5) for the end of a collision with friction.

3. We now consider the model of a collision based on (1.3) and (1.4) in the general case. To do so, we will study the second derivative of the function Z in (2.10), that is, the value of $d\Phi/dI_n$. If $\mathbf{V}_\tau \neq 0$, from (2.8) and the Bessel inequality, we obtain the following estimate:

$$\frac{d\Phi}{dI_n} = 2\left(\mathbf{b}, -\mu\frac{d\theta_\tau}{dI_n}\right) = -\frac{2\mu}{\mid \mathbf{V}_\tau \mid}[(\mathbf{b},\mathbf{b}) - (\mathbf{b},\mathbf{n})^2 - (\mathbf{b},\theta_\tau)^2] \leqslant 0 \tag{3.3}$$

There are also two possible cases of a sudden change in the value of Φ when $\mathbf{V}_\tau$ becomes zero. In the first of these, friction prevents the resumption of slip; according to (2.9), $\Phi = (\mathbf{C}^{-1}\mathbf{n}, \mathbf{n})^{-1}$. Immediately before sliding stops

$$\Phi = (\mathbf{Cn}, \mathbf{n}) - 2\mu(\mathbf{Cn}, \theta_\tau) + \mu^2(\mathbf{C}\theta_\tau, \theta_\tau)$$

so that the increment $\Delta\Phi$ is given by the formula

$$\Delta\Phi = (\mathbf{C}^{-1}\mathbf{n}, \mathbf{n})^{-1} - (\mathbf{Cn}, \mathbf{n}) + 2\mu(\mathbf{Cn}, \theta_\tau) - \mu^2(\mathbf{C}\theta_\tau, \theta_\tau) = (\mathbf{C}^{-1}\mathbf{n}, \theta_\tau)^{-1} - (\mathbf{Cn}, \mathbf{n}) + \tag{3.4}$$

$$+ \frac{(\mathbf{Cn}, \theta_\tau)^2}{(\mathbf{C}\theta_\tau, \theta_\tau)} - \left[\mu - \frac{(\mathbf{Cn}, \theta_\tau)}{(\mathbf{C}\theta_\tau, \theta_\tau)}\right]^2 (\mathbf{C}\theta_\tau, \theta_\tau) \leqslant (\mathbf{C}^{-1}\mathbf{n}, \mathbf{n})^{-1} - (\mathbf{Cn}, \mathbf{n}) + (\mathbf{C}\theta_\tau, \theta_\tau)^{-1}(\mathbf{Cn}, \theta_\tau)^2$$

If c_{ij} are the elements of the matrix **C** in a basis of vectors **n**, θ_τ and $\mathbf{n}\times\theta_\tau$, inequality (3.4) takes the form

$$\Delta\Phi = \frac{\det|\mathbf{C}|}{c_{22}c_{33}-c_{23}^2} - c_{11} + \frac{c_{12}^2}{c_{22}} = -\frac{(c_{12}c_{23}-c_{13}c_{22})^2}{c_{22}(c_{22}c_{33}-c_{23}^2)} \leqslant 0 \tag{3.5}$$

In the second case of zero $\mathbf{V}_\tau$ the friction is insufficient to prevent slip; in this case slip is resumed in the direction θ_τ^0 for which

$$\frac{d\theta_\tau}{dI_n} = 0, \quad \frac{d|\mathbf{V}_\tau|}{dI_n} > 0 \tag{3.6}$$

since, from (2.8), as $V_n \to 0$, the magnitude of $\mathbf{b}-(\mathbf{b},\mathbf{n})\mathbf{n}-(\mathbf{b},\theta_\tau)\theta_\tau$ tends to zero (see also [5]). We put

$$\mathbf{g}_1 = \mu\theta_\tau^0, \quad \mathbf{g}_2 = \mathbf{n}\times\mathbf{g}_1, \quad \mu\theta_\tau = \mathbf{g}_1\cos\xi + \mathbf{g}_2\sin\xi$$

where the angle ξ defines the direction of sliding immediately before it stops, and $\theta_\tau = \theta_\tau^0$ after sliding has been resumed. For $\Delta\Phi$, from (3.6), we have

$$\Delta\Phi = (\mathbf{n}-\mathbf{g}_1, \mathbf{C}(\mathbf{n}-\mathbf{g}_1)) - (\mathbf{n}-\mathbf{g}_1\cos\xi-\mathbf{g}_2\sin\xi,\ \mathbf{C}(\mathbf{n}-\mathbf{g}_1\cos\xi-\mathbf{g}_2\sin\xi)) = \tag{3.7}$$

$$= (\cos\xi - 1)\,[2(\mathbf{Cn},\mathbf{g}_1) - 2(\mathbf{Cg}_1,\mathbf{g}_2)\sin\xi + (1+\cos\xi)((\mathbf{Cg}_2,\mathbf{g}_2)-(\mathbf{Cg}_1,\mathbf{g}_1))] \leqslant$$

$$\leqslant 2(\cos\xi - 1)(\mathbf{g}_1\sin\xi/2 + \mathbf{g}_2\cos\xi/2,\ \mathbf{C}(\mathbf{g}_1\sin\xi/2+\mathbf{g}_2\cos\xi/2)) \leqslant 0$$

Collecting together inequalities (3.3), (3.5) and (3.7), we arrive at the following theorem.

Theorem 2. The derivative of the function $Z(I_n)$ in (2.10) is a non-increasing function of I_n.

Corollary. The model of collision based on Eqs (1.3) and (1.4) satisfies the energy criterion of correctness (3.1).

Proof of corollary. From Theorem 1, $Z(I_n)$ is continuous and increases monotonically, and $Z(0)<0$, and so it has not more than one root if $I_n>0$. If Z_n is negative in the segment $[0, I_2]$, then $\Delta T<0$, and (3.1) is satisfied. If $Z(I_0)=0$ for a certain value $I_0<I_2$, the integration interval in (2.1) can be divided into two parts: from 0 to I_0 and from I_0 to I_2. Since $Z=V_n-\mu|\mathbf{V}_\tau|$, $Z(I_1)\leqslant 0$ (I_1 is the value corresponding to the end of Newtonian deformation), $V_n(I_1)=0$ and, therefore, $I_1\leqslant I_0$ and

$$\frac{I_2-I_0}{I_0} \leqslant \frac{I_2-I_1}{I_1} = \kappa < 1 \tag{3.8}$$

which shows that the interval in which Z is positive is smaller than that in which it is negative (within the integration segment). From Theorem 2 it follows that

$$Z(I_0+x) \leqslant -Z(I_0-x), \quad 0<x<I_0$$

from which the estimate (3.1) for ΔT in (2.10) follows for $\kappa\leqslant 1$.

We note that the reason for the incorrectness of the model based on (1.5) is that it does not, in general, follow from the inequality $e<1$ in (1.5) that κ is less than one in (1.4) and in (3.8). In particular, for the example given before, we have calculated $I_0=-0.57(V_n^-+5.25I^*)=0.31$, from which we find $\kappa=1.48>1$.

4. ENERGY APPROACH TO DETERMINATION OF THE COEFFICIENT OF RESTITUTION

Although the kinetic model of the coefficient of restitution (1.5) is correct, an energy approach to the two collision phases and the coefficient of restitution is more valid in physical terms. In the first of the phases, the elastic strain potential energy accumulates, and is then released during recovery.

In the collision of solids with smooth surfaces ($\mu=0$), the energy changes in the two collision phases can be expressed by Eq. [1].

$$\eta^2 = \frac{T^+ - T_0}{T^- - T_0} \tag{4.1}$$

where T_0 is the lowest value of the kinetic energy during the collision, equal to its value at the end of the first phase, and the coefficient of restitution η is identical in this case with the coefficients e and κ.

Formula (4.1) can be taken as the direct, energy definition of the coefficient of restitution η.

An extension of (4.1) to the case of collision with friction of the form (1.3) has been proposed in [8], on the assumption that there is no tangential compliance of the colliding solids. In that case, (2.2) becomes

$$\Delta T = \int_0^{I_2} V_n dI_n + \int_{0I_\tau^*} \mathbf{V}_\tau d\mathbf{I}_\tau \tag{4.2}$$

where the second term expresses dissipation due to irreversible strains during friction (external dissipation), and the first expresses the work of the normal reaction. With the given notation, the coefficient of restitution η_* is defined as

$$\eta_*^2 = -\int_{I_1}^{I_2} V_n dI_n \Big/ \int_0^{I_1} V_n dI_n \tag{4.3}$$

We will examine the laws of change of the energy in those cases where the effect of tangential compliance is substantial (see [9, 10]). In that case, the tangential strains during the collision are not irreversible and dissipation is due only to internal processes in the colliding solids. This hypothesis is obviously inconsistent with the tangential stresses defined by (1.3): if $\mathbf{V}_\tau$ changes direction during the collision, these stresses depend on the relative displacement at the point A. We therefore restrict ourselves to the special case where the directions of $\mathbf{V}_\tau$ and the impact reaction are unchanged during the collision.

We will represent the impact impulse in the form

$$\mathbf{I} = \alpha \mathbf{I}', \quad \mathbf{I}' = \mathbf{n} - \mu\theta_\tau^-, \quad \alpha > 0 \tag{4.4}$$

and use Kelvin's formula (2.5). Since

$$\Delta T = ½(\mathbf{V}^- + \mathbf{V}^+, \alpha\mathbf{I}') = \alpha(\mathbf{V}^-, \mathbf{I}') = \alpha(\mathbf{V}^-, \mathbf{I}') + ½\alpha^2(\mathbf{I}', C\mathbf{I}') \tag{4.5}$$

the minimum energy T_0 is attained for the value $\alpha = \alpha_0$, where

$$\alpha_0 = -\frac{(\mathbf{V}^-, \mathbf{I}')}{(C\mathbf{I}', \mathbf{I}')}, \quad \Delta T_0 = T_0 - T^- = -\frac{1}{2}\frac{(\mathbf{V}^-, \mathbf{I}')^2}{(C\mathbf{I}', \mathbf{I}')} \tag{4.6}$$

If the total impact impulse is equal to $\mathbf{I}^* = \alpha^2\mathbf{I}'$, from (2.5) we have

$$\Delta T = ½(\mathbf{V}^- + \mathbf{V}^+, \alpha_2\mathbf{I}') = (½\alpha_2^2 - \alpha_0\alpha_2)(C\mathbf{I}', \mathbf{I}') \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.1), we obtain for the coefficient of restitution η

$$\eta = \frac{\alpha_2 - \alpha_0}{\alpha_0} = -\frac{(\mathbf{V}^+, \mathbf{I}')}{(\mathbf{V}^-, \mathbf{I}')} \tag{4.8}$$

We have thus proved the following theorem.

Theorem 3. If the reaction does not change direction during the collision, the minimum value of the kinetic energy T_0 in (4.1) corresponds to disappearance of the projection of the relative velocity at the point of contact in that direction. In this case, η is equal to the ratio of the projections of the relative velocity on that direction, taken with the opposite sign.

Note that the definition of the coefficient of restitution (4.8) was used in [11] in the analysis of automobile collision: this coefficient is better from a practical standpoint than either e or κ. It was assumed there that η can take values between -1 and 1: negative values of η correspond to cases where the kinetic energy decreases over the course of the entire collision. The other limiting case $h = 1$ corresponds to absolutely elastic collision of absolutely rough solids (see [12]).

Note that the correctness criterion (3.1) is satisfied automatically with the energy definitions of the coefficient of restitution.

5. ANALYSIS OF EXPERIMENTAL RESULTS

The experimental results of investigations of the collision of a spherical particle on a rough surface are given in [8, 13–15]. In that case, the direction of the reaction during the collision can be taken to be unchanged and definitions (1.4), (1.5) and (4.3) are equivalent: $\kappa = e = \eta^*$, but are different from (4.1). One result is that the V_n^+ increases with V_τ^- for a fixed value of V_n^-. Thus, an increase in the value of e from 0.45 to 0.63 as V_τ^- changes from 0 to 5 (m/s) with $V_n^- = -1$ m/s was noted in [13]. The model (1.4) is obviously unsuitable for describing this effect. At the same time, from formula (4.8) we obtain the following dependence of the coefficient e on the angle of attack φ

$$e = \frac{1}{\cos\varphi}\left[\sin(\varphi-\beta)\sin\beta + \eta\cos(\varphi-\beta)\cos\beta\right] = \frac{|V_\tau^-|}{|V_n^-|}(1+\eta)\frac{\mu}{1+\mu^2} + \frac{\eta-\mu^2}{1+\mu^2}, \quad \beta = \operatorname{arctg}\mu \tag{5.1}$$

that is, e increases linearly with $|V_\tau^-|$.

Results on the collision of a steel sphere of 1 mm diameter and a steel plate at an initial velocity of 10 (m/s) and angles of attack $\varphi_1 = 0$, $\varphi_2 = 45°$ and $\varphi_3 = 75°$ are given in [14, 15]. For the loss coefficient $\zeta = -\Delta T/T^-$ here we obtained the values $\zeta_1 = 0.53$, $\zeta_2 = 0.32$ and $\zeta_3 = 0.07$, and for the coefficient of impact friction $\mu = 0.12$.

The coefficient $e = \kappa = \eta_*$ can be calculated in terms of ζ and μ from the formula

$$1 - \zeta = \kappa^2\cos^2\varphi + [\sin\varphi - \mu(1+\kappa)\cos\varphi]^2 \tag{5.2}$$

giving

$$\kappa_1 = 0.69, \quad \kappa_2 = 0.87, \quad \kappa_3 = 1.49$$

Note that the value of κ_3 does not even reach the range of values allowed in (1.4).

The coefficient η can be calculated from the relation:

$$1 - \zeta = \frac{(\sin\varphi - \mu\cos\varphi)^2 + \eta^2(\cos\varphi + \mu\sin\varphi)^2}{1+\mu^2}$$

giving

$$\eta_1 = 0.69, \quad \eta_2 = 0.69, \quad \eta_3 = 0.71.$$

Consequently, in this case the definition (4.1) is more realistic than the model of collision with friction based on (1.4), (1.5) or (4.3).

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THE SEPARATRIX OF AN UNSTABLE POSITION OF EQUILIBRIUM OF A HESS–APPELROT GYROSCOPE†

S. A. DOVBYSH

Moscow

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The motion of a heavy solid with a fixed point whose inertial tensor and the centre of mass (which does not coincide with the point of support) satisfy the Hess–Appelrot (HA) conditions is considered. At the zeroth value of the areas constant, the gyroscope has an unstable position of equilibrium at which the radius vector drawn from the point of support to the centre of mass is directed vertically upwards. Solutions which are asymptotic to this position of equilibrium form two-dimensional ingoing and outgoing separatrices which satisfy the Hess conditions and are therefore identical (they are paired). The motion close to a paired separatrix is considered (when, generally speaking, the particular Hess integral may be non-zero) and families of long-period solutions are found. Splitting of the separatrices when an HA gyroscope is perturbed is studied. The results obtained are used to investigate the separatrices of a perturbed Lagrange problem for a value of the areas constant close to zero. In particular, the occurrence of double-detour homoclinic solutions, which leads to the non-integrability of the problem, is demonstrated in the case of a zero value for the areas constant. The occurrence of single-detour homoclinic solutions of the perturbed Lagrange problem, leading to non-integrability for non-zero values of the areas constant has previously been found in [1].

1. FORMULATION OF THE PROBLEM

IN A NUMBER of cases it is convenient to make use of a special system of coordinates [2, 3] in the study of a solid with a fixed point, that is, a Cartesian system of coordinates which is rigidly fixed in the body where the unit vector directed from the point of support to the centre of gravity has the

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